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A Bound for the Remainder of the Hilbert-Schmidt Series and Other Results on Representation of Solutions to the Functional Equation of the Second Kind with a Self-Adjoint Compact Operator as an Infinite Series

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Abstract—For the functional equation of the second kind (see (1)) $\phi - \lambda \mathbf{K} \phi = \mathbf{f}$, with $\mathbf{K}$ a compact self-adjoint linear operator on a Hilbert space (a Fredholm integral equation of the second kind, for example), a bound for the remainder of the Hilbert-Schmidt series is found. It is shown that the series solution to (1) introduced in the author's previous paper [1] is (much) more rapidly convergent than the Hilbert-Schmidt series and generally speaking, is a preferable way of expressing the solution to (1) for regular λ as an infinite series. Other series solutions to (1) are given. The corresponding expressions for the inverse $(\mathbf{I} - \lambda \mathbf{K})^{-1}$ and the resolvent $\mathbf{B}_\lambda$, and also for the resolvent of the Fredholm integral equation of the second kind with symmetric kernel, are given too.

Keywords—Hilbert-Schmidt series, Equation of the second kind, Self-adjoint operator, Series solution, Remainder, Bound.

1. INTRODUCTION

The functional equation of the second kind [2]

$$\phi - \lambda \mathbf{K} \phi = \mathbf{f}, \tag{1}$$

where $\mathbf{f}$ and ϕ are, respectively, known and unknown elements of a Hilbert space H, $\mathbf{K}$ is a compact self-adjoint linear operator on H, and λ is a real or complex parameter, arises often in the mathematical treatment of problems of physics and engineering. One special case of (1) which is important in applications is a Fredholm integral equation of the second kind [2,3].

Let $\{\lambda_j\}$ and $\{\phi_j\}$ be the system of characteristic values and characteristic vectors of $\mathbf{K}$: all λ_j are real; the set $\{\lambda_j\}$ is countable (or finite, in special cases); characteristic values are repeated in the sequence $\{\lambda_j\}$ according to their multiplicities; $0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_j| \le \cdots$; only one ϕ_j (of $\{\phi_j\}$) corresponds to a λ_j (of $\{\lambda_j\}$); and the system $\{\phi_j\}$ is orthonormal [2,3].

For any regular λ, $\lambda \ne \lambda_j$, for $j = 1, 2, \dots$, the solution ϕ to (1) (with a self-adjoint $\mathbf{K}$) exists and is unique; the solution can be represented in the form of the Hilbert-Schmidt series [2,3]

$$\phi = \mathbf{f} - \lambda \sum_{j=1}^{\infty} \frac{(\mathbf{f}, \phi_j)\, \phi_j}{\lambda - \lambda_j}, \tag{2}$$

which converges in H for any $\mathbf{f} \in H$.

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Even in the case when all λ_j and ϕ_j are known, it is impossible to take into account an infinite number of terms in (2), in performing numerical calculations. It is of importance, therefore, to have an estimate for the remainder if a finite number of terms are retained. It is especially important in the case when only a finite number of λ_j and ϕ_j are known (and the rest of them are unknown).

The author did not find any estimates for the remainder of (2) in the literature. In this paper, an estimate for the remainder of (2) is found.

In [1], a different series representation of the solution to (1) for regular λ was found, namely,

$$\phi = \sum_{n=0}^{N} \lambda^n \mathbf{K}^n \mathbf{f} - \lambda \sum_{j=1}^{\infty} \left(\frac{\lambda}{\lambda_j}\right)^N \frac{(\mathbf{f}, \phi_j)\,\phi_j}{\lambda - \lambda_j}, \qquad N = 0, 1, 2, \dots. \tag{3}$$

The estimate for the remainder of (3) is also found in this paper; the estimate suggests that the infinite series of (3) is much more rapidly convergent than the infinite series of (2) (and the more rapidly, the larger is N). This means that in constructing the infinite series solution to (1), valid for regular λ, it may be to one's benefit to construct it in the form (3) (with a small positive N ($N = 1, 2$, or 3, say))—instead of the Hilbert-Schmidt series (2).

Other results cited in the paper are: one more (different from (2) and (3)) representation of the solution to (1) for regular λ as an infinite series; a representation of the solution to (1) for regular λ on disks $|\lambda| < |\lambda_{m+1}|$ as an infinite series; a representation of solutions to (1) as an infinite series, in the case when λ is a characteristic value (provided that such solutions exist); certain expressions for the inverse operator $(\mathbf{I} - \lambda\mathbf{K})^{-1}$ and the resolvent $\mathbf{B}_\lambda$ [2], and certain expressions for the resolvent of the Fredholm integral equation of the second kind with symmetric kernel, as an infinite series. One of these results is used in obtaining the estimate for the remainder of (2); others make the account of the subject of the paper more complete, and—the author entertains a hope—can be used to obtain new results in the direction of this paper.

For a number of the results obtained in the paper, their usefulness in finding numerical solutions to (1) is self-evident. In one case (namely, with respect to Equation (26)), detailed explanations regarding this matter are given.

2. BOUND FOR THE REMAINDER OF THE HILBERT-SCHMIDT SERIES

THEOREM. *Let $\lambda_1, \dots, \lambda_m, \lambda_{m+1}$ be the first $m+1$ ($m \geq 1$) characteristic values of a compact self-adjoint linear operator $\mathbf{K}$ on a Hilbert space H, and let $\lambda_m \neq \lambda_{m+1}$. Then, for any regular λ on the disk $|\lambda| < |\lambda_{m+1}|$, the following estimate for the remainder of* (2),

$$\mathbf{R}_H = -\lambda \sum_{j=m+1}^{\infty} \frac{(\mathbf{f}, \phi_j)\,\phi_j}{\lambda - \lambda_j}, \tag{4}$$

is true:

$$\|\mathbf{R}_H\| \leq \left\| \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\,\phi_k \right\| \frac{|\lambda/\lambda_{m+1}|}{1 - |\lambda/\lambda_{m+1}|}. \tag{5}$$

Below, we shall prove this theorem. But before doing so, we shall find a different series solution to (1) valid for regular λ.

3. ANOTHER SERIES SOLUTION TO EQUATION (1) VALID FOR REGULAR λ

Let us rewrite (1), for regular λ, as

$$\phi^* - \lambda\,\mathbf{K}\,\phi^* = \mathbf{f}^*, \tag{6}$$

where

$$\boldsymbol{\phi}^* = \boldsymbol{\phi} + \sum_{k=1}^{m} \frac{(\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k\, \lambda_k}{\lambda - \lambda_k}, \qquad \text{and} \tag{7}$$

$$\mathbf{f}^* = \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k. \tag{8}$$

Writing out the solution to (6) in the form (3), with $\boldsymbol{\phi}$ and $\mathbf{f}$ of (3) replaced by $\boldsymbol{\phi}^*$ and $\mathbf{f}^*$, respectively, one finds the following solution to (1):

$$\boldsymbol{\phi} = \mathbf{f} - \lambda \sum_{k=1}^{m} \frac{(\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k}{\lambda - \lambda_k} + \sum_{n=1}^{N} \lambda^n \mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right] - \lambda \sum_{j=m+1}^{\infty} \left(\frac{\lambda}{\lambda_j} \right)^N \frac{(\mathbf{f}, \boldsymbol{\phi}_j)\, \boldsymbol{\phi}_j}{\lambda - \lambda_j}, \tag{9}$$

for regular λ and $N = 1, 2, \dots, m = 1, 2, \dots$.

4. PROOF OF THE THEOREM

Comparing (2) to (9), we find

$$\mathbf{R}_H = \sum_{n=1}^{N} \lambda^n \mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right] - \lambda \sum_{j=m+1}^{\infty} \left(\frac{\lambda}{\lambda_j} \right)^N \frac{(\mathbf{f}, \boldsymbol{\phi}_j)\, \boldsymbol{\phi}_j}{\lambda - \lambda_j}, \qquad N = 1, 2, \dots. \tag{10}$$

Consider now the operator (self-adjoint, compact [2])

$$\mathbf{K}^{(m)} = \mathbf{K} - \sum_{k=1}^{m} \lambda_k^{-1}\, \mathbf{P}_k, \tag{11}$$

where $\mathbf{P}_k$ is the projector onto the one-dimensional subspace corresponding to λ_k. In [2], it is shown that $\mathbf{K}\,\mathbf{P}_k = \mathbf{P}_k\,\mathbf{K}$; in a similar manner it is possible to show that $\mathbf{K}^n\,\mathbf{P}_k = \mathbf{P}_k\,\mathbf{K}^n$ for any positive integer n. Using that, we find

$$\mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right] = \left[\mathbf{K}^{(m)} \right]^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right], \qquad n = 0, 1, 2, \dots. \tag{12}$$

Taking into account that [2]

$$\left\| \mathbf{K}^{(m)} \right\| = |\lambda_{m+1}|^{-1}, \tag{13}$$

one finds, from (10), that

$$\|\mathbf{R}_H\| \le \left\| \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right\| \sum_{n=1}^{N} \left| \frac{\lambda}{\lambda_{m+1}} \right|^n + \left| \frac{\lambda}{\lambda_{m+1}} \right|^N \left[|\lambda|^2 \sum_{j=m+1}^{\infty} \frac{|(\mathbf{f}, \boldsymbol{\phi}_j)|^2}{|\lambda - \lambda_j|^2} \right]^{1/2}, \tag{14}$$

and then

$$\|\mathbf{R}_H\| \left(1 - \left| \frac{\lambda}{\lambda_{m+1}} \right|^N \right) \le \left\| \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \boldsymbol{\phi}_k)\, \boldsymbol{\phi}_k \right\| \left| \frac{\lambda}{\lambda_{m+1}} \right| \frac{1 - |\lambda/\lambda_{m+1}|^N}{1 - |\lambda/\lambda_{m+1}|}, \tag{15}$$

for $N = 1, 2, \dots$. For $|\lambda| < |\lambda_{m+1}|$, this gives us the estimate (5) and the theorem is proved.

5. BOUND FOR THE REMAINDER OF EQUATION (3)

For the remainder of (3),

$$\mathbf{R}_T = -\lambda \sum_{j=m+1}^{\infty} \left(\frac{\lambda}{\lambda_j}\right)^N \frac{(\mathbf{f}, \phi_j)\, \phi_j}{\lambda - \lambda_j}, \tag{16}$$

we have

$$\|\mathbf{R}_T\| \le |\lambda/\lambda_{m+1}|^N \, \|\mathbf{R}_H\|, \tag{17}$$

which gives us, instead of (5), the following bound:

$$\|\mathbf{R}_T\| \le \left\| \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k \right\| \frac{|\lambda/\lambda_{m+1}|^{N+1}}{1 - |\lambda/\lambda_{m+1}|}, \qquad \text{for regular } \lambda,\ |\lambda| < |\lambda_{m+1}|. \tag{18}$$

In (5) and (18),

$$\left\| \mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k \right\| \underset{m\to\infty}{\longrightarrow} \|\mathbf{f}_0\|, \tag{19}$$

where $\mathbf{f}_0$ is the component of $\mathbf{f}$ orthogonal to $\{\phi_j\}$ ($\mathbf{f}_0 \perp \overline{\mathbf{K}(H)}$). The multiplicative factors in the bounds (5) and (18) tend to zero as $m \to \infty$ (much more rapidly in (18), due to the presence of $|\lambda/\lambda_{m+1}|^{N+1}$ instead of $|\lambda/\lambda_{m+1}|^1$). The bound (18) suggests that the infinite series (3) is, generally speaking, much more rapidly convergent than the series (2) (and the more rapidly the larger is N).

6. EXAMPLE ON CONSTRUCTING SOLUTIONS (2) AND (3)

Consider the integral equation [4,5]

$$\phi(x) - \lambda \int_0^{1/2} K(x,s)\, \phi(s)\, ds = 1, \tag{20}$$

where

$$K(x,s) = \begin{cases} x, & \text{if } s \ge x; \\ s, & \text{if } s < x. \end{cases} \tag{21}$$

This equation is a particular case of the equation which arises in the theory of an inviscid, incompressible, weightless thin jet flowing from a nozzle onto the surface of a heavy liquid [4]. In this case,

$$K^n f = \frac{(-1)^n}{(2n)!} E_{2n}(x), \qquad n = 1, 2, \dots, \tag{22}$$

where $E_{2n}(x)$ are the Euler polynomials and [5]

$$\lambda_j = [(2j-1)\,\pi]^2, \quad \phi_j = 2\,\sin(2j-1)\pi x, \qquad j = 1, 2, \dots. \tag{23}$$

The Hilbert-Schmidt solution (2) to (20) is

$$\phi(x) = 1 + \frac{4\lambda}{\pi} \sum_{j=1}^{\infty} \frac{1}{(2j-1)} \frac{\sin(2j-1)\pi x}{[(2j-1)\pi]^2 - \lambda}, \tag{24}$$

and the solution in the form (3) is

$$\phi(x) = 1 + \sum_{n=1}^{N} \lambda^n \frac{(-1)^n}{(2n)!} E_{2n}(x) + \frac{4\lambda}{\pi} \left(\frac{\lambda}{\pi^2}\right)^N \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2N+1}} \frac{\sin(2j-1)\pi x}{[(2j-1)\pi]^2 - \lambda}, \tag{25}$$

for $N = 1, 2, \dots$. The factors $(2j-1)^{-(2N+1)}$ contribute greatly toward the increase in rapidity of convergence of the infinite series of (25)—comparing to the infinite series of the Hilbert-Schmidt solution (24). Thus, using solution (3) to (1), we were able to represent the solution to the integral equation (20) in the form of the series converging more rapidly than if we used the Hilbert-Schmidt solution (2) to (1) for that purpose.

7. OTHER RESULTS ON REPRESENTATION OF SOLUTIONS TO EQUATION (1) AS AN INFINITE SERIES

Now we shall cite other similar results on the representation of solutions to (1) (with a self-adjoint compact $\mathbf{K}$) as an infinite series, namely:

(i) *If* $|\lambda_m| < |\lambda_{m+1}|$, *then for regular* λ *on the disk* $|\lambda| < |\lambda_{m+1}|$,

$$\phi = \mathbf{f} - \lambda \sum_{k=1}^{m} \frac{(\mathbf{f}, \phi_k)\, \phi_k}{\lambda - \lambda_k} + \sum_{n=1}^{\infty} \lambda^n \mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k \right]. \tag{26}$$

To derive (26), we first consider the equation

$$\phi - \lambda \mathbf{K} \phi = \tilde{\mathbf{f}}, \tag{27}$$

in the case when $|\lambda_m| < |\lambda_{m+1}|$ and

$$\left(\tilde{\mathbf{f}}, \phi_k\right) = 0, \qquad \text{for } k = 1, 2, \dots, m. \tag{28}$$

The Neumann series solution to (27) is

$$\phi = \sum_{n=0}^{\infty} \lambda^n \mathbf{K}^n \tilde{\mathbf{f}}, \tag{29}$$

and, taking into account (12), Equation (29) can be rewritten as

$$\phi = \sum_{n=0}^{\infty} \lambda^n \left[\mathbf{K}^{(m)}\right]^n \tilde{\mathbf{f}}. \tag{30}$$

From (30) and (13), it follows that the Neumann series (29) converges on the disk $|\lambda| < |\lambda_{m+1}|$; it represents the solution to (27) on that disk for regular λ. Note that if λ is the characteristic value

$$\lambda = \lambda_p = \lambda_{p+1} = \cdots = \lambda_q, \tag{31}$$

with $p \geq 1,\ q \leq m$, then the solution to (27) is

$$\phi = \sum_{n=0}^{\infty} \lambda_p^n \mathbf{K}^n \tilde{\mathbf{f}} + \sum_{j=p}^{q} a_j\, \phi_j. \tag{32}$$

Here and below, a_j are arbitrary constants.

We now rewrite (1) (for regular λ) as (6), and then we use (29) to find that

$$\phi^* = \sum_{n=0}^{\infty} \lambda^n \mathbf{K}^n \mathbf{f}^*, \tag{33}$$

for regular λ on the disk $|\lambda| < |\lambda_{m+1}|$. From (33), (7), and (8), Equation (26) follows.

(ii) *If* λ *is the characteristic value* (31) *with* $p \geq 1$, $q \leq m$, *and if the conditions*

$$(\mathbf{f}, \phi_\ell) = 0, \qquad \textit{for } \ell = p, p+1, \dots, q, \tag{34}$$

are satisfied, then

$$\phi = \mathbf{f} - \lambda_p {\sum_{k=1}^{m}}' \frac{(\mathbf{f}, \phi_k)\, \phi_k}{\lambda_p - \lambda_k} + \sum_{n=1}^{\infty} \lambda_p^n \mathbf{K}^n \left[\mathbf{f} - {\sum_{k=1}^{m}}' (\mathbf{f}, \phi_k)\, \phi_k \right] + \sum_{j=p}^{q} a_j\, \phi_j. \tag{35}$$

Here and below, $\sum'$ means that values of the parameter of summation equal to p, $p+1, \dots, q$ should be omitted.

If λ is the characteristic value (31), and the conditions (34) are satisfied, then

$$\phi = \sum_{n=0}^{N} \lambda_p^n \mathbf{K}^n \mathbf{f} - \lambda_p \sum_{j=1}^{\infty}{}' \left(\frac{\lambda_p}{\lambda_j}\right)^N \frac{(\mathbf{f}, \phi_j)\, \phi_j}{\lambda_p - \lambda_j} + \sum_{j=p}^{q} a_j\, \phi_j, \tag{36}$$

for $N = 0, 1, 2, \dots$.

If λ is the characteristic value (31), with $p > 1$, and the conditions (34) are satisfied, then

$$\phi = \mathbf{f} - \lambda_p \sum_{k=1}^{m} \frac{(\mathbf{f}, \phi_k)\, \phi_k}{\lambda_p - \lambda_k} + \sum_{n=1}^{N} \lambda_p^n \mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k\right] - \lambda_p \sum_{j=m+1}^{\infty}{}' \left(\frac{\lambda_p}{\lambda_j}\right)^N \frac{(\mathbf{f}, \phi_j)\, \phi_j}{\lambda_p - \lambda_j} + \sum_{j=p}^{q} a_j\, \phi_j, \tag{37}$$

where $m = 1, 2, \dots, p-1$, $N = 1, 2, \dots$.

The solutions (26), (9), and (37) can be written somewhat differently if we take into account that

$$\mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k\right] = \mathbf{K}^n \mathbf{f} - \sum_{k=1}^{m} \frac{(\mathbf{f}, \phi_k)\, \phi_k}{\lambda_k^n}, \tag{38}$$

and similarly for (35).

The solution to (1) in the form (26) can be used in applications. Consider the following situation, for example: one needs to have a way of finding solutions of the Fredholm integral equation of the second kind with symmetric kernel for λ on the interval between the first and the second characteristic values of the kernel, that is, for $\lambda \in (\lambda_1, \lambda_2)$, in the case when $0 < \lambda_1 < \lambda_2 < \lambda_3$. The exact values of λ for which the solutions should be found are not known in advance; the selection is to be made by means of complicated numerical calculations to satisfy certain conditions of an engineering (physical) problem the integral equation in question describes. (A problem of this type arises in the hydrodynamic theory of air-cushioned vehicles [4].)

One way to solve the problem is to use the solution to the integral equation in the form (26): to find λ_1, λ_2, and λ_3, and ϕ_1, ϕ_2, and ϕ_3, say, to construct (26) (with certain number of terms in the sum on n), and then calculate the solutions to the integral equation in question for any $\lambda \in (\lambda_1, \lambda_2)$, by just substituting different λ in (26).

One advantage of this approach can be that our calculations should not, generally speaking, be sensitive to λ being close to λ_1 or to λ_2—since the singularities at λ_1, λ_2 of the solution are already extracted in (26) (by the sum on k; but of course, we should calculate λ_1 and λ_2 and ϕ_1 and ϕ_2 with the proper accuracy).

As an example of the solution to (1) in the form of (26), we give here the solution to the integral equation (20) in the form (26), with $m = 1$:

$$\phi(x) = 1 - \frac{4\lambda \sin \pi x}{\pi(\lambda - \pi^2)} + \sum_{n=1}^{\infty} \lambda^n \left[\frac{(-1)^n}{(2n)!} E_{2n}(x) - \frac{4 \sin \pi x}{\pi^{2n+1}}\right], \qquad |\lambda| < 9\pi^2,\ \lambda \neq \pi^2. \tag{39}$$

For the remainder of (26),

$$\mathbf{R}_D = \sum_{n=\nu+1}^{\infty} \lambda^n \mathbf{K}^n \left[\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k\right], \tag{40}$$

the following estimate is true:

$$\|\mathbf{R}_D\| \le \left\|\mathbf{f} - \sum_{k=1}^{m} (\mathbf{f}, \phi_k)\, \phi_k\right\| \frac{|\lambda/\lambda_{m+1}|^{\nu+1}}{1 - |\lambda/\lambda_{m+1}|}, \qquad \nu = 0, 1, 2, \dots. \tag{41}$$

8. THE INVERSE $(\mathbf{I}-\lambda\mathbf{K})^{-1}$ AND THE RESOLVENT $\mathbf{B}_\lambda$

Now, we shall rewrite (26), (3), and (9) in terms of operators $\mathbf{P}_k$ and $\mathbf{K}^{(m)}$, and give the corresponding expressions for the inverse $(\mathbf{I}-\lambda\mathbf{K})^{-1}$ and for the resolvent $\mathbf{B}_\lambda$ [2]. From (26),

$$\phi = \left\{\mathbf{I} - \lambda\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{\infty}\lambda^n\left[\mathbf{K}^{(m)}\right]^n\right\}\mathbf{f}, \tag{42}$$

$$(\mathbf{I}-\lambda\mathbf{K})^{-1} = \mathbf{I} - \lambda\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{\infty}\lambda^n\left[\mathbf{K}^{(m)}\right]^n, \quad \text{and} \tag{43}$$

$$\mathbf{B}_\lambda = \lambda^{-1}\left[(\mathbf{I}-\lambda\mathbf{K})^{-1}-\mathbf{I}\right] = -\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{\infty}\lambda^{n-1}\left[\mathbf{K}^{(m)}\right]^n, \tag{44}$$

for regular λ on the disk $|\lambda| < |\lambda_{m+1}|$, provided that $|\lambda_m| < |\lambda_{m+1}|$. From (3),

$$\phi = \left\{\sum_{n=0}^{N}\lambda^n\mathbf{K}^n - \lambda\sum_{j=1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}\right\}\mathbf{f}, \tag{45}$$

$$(\mathbf{I}-\lambda\mathbf{K})^{-1} = \sum_{n=0}^{N}\lambda^n\mathbf{K}^n - \lambda\sum_{j=1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}, \tag{46}$$

for regular λ, $N = 0, 1, 2, \ldots,$

$$\mathbf{B}_\lambda = -\sum_{j=1}^{\infty}\frac{\mathbf{P}_j}{\lambda-\lambda_j}, \quad \text{and then} \tag{47}$$

$$\mathbf{B}_\lambda = \sum_{n=1}^{N}\lambda^{n-1}\mathbf{K}^n - \sum_{j=1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}, \quad \text{for regular } \lambda,\ N = 1, 2, \ldots. \tag{48}$$

Finally, from (9),

$$\phi = \left\{\mathbf{I} - \lambda\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{N}\lambda^n\left[\mathbf{K}^{(m)}\right]^n - \lambda\sum_{j=m+1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}\right\}\mathbf{f}, \tag{49}$$

$$(\mathbf{I}-\lambda\mathbf{K})^{-1} = \mathbf{I} - \lambda\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{N}\lambda^n\left[\mathbf{K}^{(m)}\right]^n - \lambda\sum_{j=m+1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}, \tag{50}$$

$$\mathbf{B}_\lambda = -\sum_{k=1}^{m}\frac{\mathbf{P}_k}{\lambda-\lambda_k} + \sum_{n=1}^{N}\lambda^{n-1}\left[\mathbf{K}^{(m)}\right]^n - \sum_{j=m+1}^{\infty}\left(\frac{\lambda}{\lambda_j}\right)^N\frac{\mathbf{P}_j}{\lambda-\lambda_j}, \tag{51}$$

for regular λ, $m = 1, 2, \ldots,$ $N = 1, 2, \ldots.$

9. SERIES REPRESENTATION FOR RESOLVENT OF THE FREDHOLM INTEGRAL EQUATION

In this section, we write out (without derivation) expressions for the resolvent $\Gamma(x, s; \lambda)$ corresponding to (26), (3), and (9) for the Fredholm integral equation of the second kind with symmetric kernel $K(x,s) = \overline{K(s,x)}$:

$$\phi(x) - \lambda K\phi = f(x), \qquad K\phi = \int_a^b K(x,s)\,\phi(s)\,ds, \tag{52}$$

where $f \in L^2(a,b)$ and $K(x,s) \in L^2$ $(a < x,\ s < b)$ [3]:

$$\Gamma(x,s;\lambda) = -\sum_{k=1}^{m} \frac{\phi_k(x)\,\overline{\phi_k(s)}}{\lambda - \lambda_k} + \sum_{n=1}^{\infty} \lambda^{n-1} \left[K_n(x,s) - \sum_{k=1}^{m} \frac{\phi_k(x)\,\overline{\phi_k(s)}}{\lambda_k^n} \right], \tag{53}$$

for regular λ, $|\lambda| < |\lambda_{m+1}|$, provided that $|\lambda_m| < |\lambda_{m+1}|$;

$$\Gamma(x,s;\lambda) = \sum_{n=1}^{N} \lambda^{n-1} K_n(x,s) - \sum_{j=1}^{\infty} \left(\frac{\lambda}{\lambda_j}\right)^N \frac{\phi_j(x)\,\overline{\phi_j(s)}}{\lambda - \lambda_j}, \tag{54}$$

for regular λ, $N = 1, 2, \dots$; and

$$\begin{aligned}\Gamma(x,s;\lambda) = -\sum_{k=1}^{m} \frac{\phi_k(x)\,\overline{\phi_k(s)}}{\lambda - \lambda_k} + \sum_{n=1}^{N} \lambda^{n-1} \left[K_n(x,s) - \sum_{k=1}^{m} \frac{\phi_k(x)\,\overline{\phi_k(s)}}{\lambda_k^n} \right] \\ - \sum_{j=m+1}^{\infty} \left(\frac{\lambda}{\lambda_j}\right)^N \frac{\phi_j(x)\,\overline{\phi_j(s)}}{\lambda - \lambda_j},\end{aligned} \tag{55}$$

for regular λ, $m = 1, 2, \dots,\ N = 1, 2, \dots$.

10. FINAL NOTE

The results (1), (3), (9), (11), (26), (31), (34)–(38), (44), (48), and (51)–(55) of this paper are cited (without proof) in abstracts [6–9]. The results (1), (3), (9), (26), (46), (48), and (52) were cited (and the rest of the results just listed were mentioned) in the author's presentation at the 100^{th} Annual Meeting of the American Mathematical Society in Cincinnati, Ohio, on January 12, 1994.

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